

Some applications of the chromatic polynomials

Mohammed Said Maamra and Miloud Mihoubi

Faculty of Mathematics, RECITS Laboratory, DG-RSDT
USTHB, BP 32, El-Alia, 16111, Algeries, Algeria
mmaamra@usthb.dz mmihoubi@usthb.dz

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Abstract

The chromatic polynomials are studied by several authors and have important applications in different frameworks, specially, in graph theory and enumerative combinatorics. The aim of this work is to establish some properties of the coefficients of the chromatic polynomial of a graph. Three applications on restricted Stirling numbers of the second kind are given.

Keywords: Chromatic polynomial of a graph; Stirling numbers; log-concavity; Pólya-frequency sequences.

Mathematics Subject Classifications: 05C15; 05C69; 11B73

1 Introduction

The chromatic polynomial was introduced by Birkhoff [1] and studied later by Whitney [17, 18], Birkhoff and Lewis [2], Read [11] and several other authors. The chromatic polynomial of a graph can be used as a tool to find the number of possible partitions of a finite set under some particular restraints such that the Stirling numbers of the second kind [16]. For a given graph $G = (V, E)$ of order n and $\lambda \in \mathbb{N}$, a mapping $f : V \rightarrow \{1, 2, \dots, \lambda\}$ is called a λ -coloring of G if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G . The λ -colorings f and g of G are regarded as distinct if $f(x) \neq g(x)$ for some x in G . The chromatic polynomial $P(G, \lambda)$ counts the number of (proper) λ -colorings of G . For example, it is known that $P(O_n, \lambda) = \lambda^n$, $P(K_n, \lambda) = (\lambda)_n$ and $P(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}$, $n \geq 1$, where O_n is a graph of order n and without edges, K_n is the complete graph of order n , T_n is a tree of order n and $(\lambda)_n = \lambda(\lambda - 1) \cdots (\lambda - n + 1)$ if $n \geq 1$ and $(\lambda)_0 = 1$. More generally, the chromatic polynomial of G can be written as $P(G, \lambda) = \sum_{i=\chi(G)}^n \alpha_i(G) (\lambda)_i$, see [4, Thm. 1.4.1], where $\alpha_i(G)$ is

the number of ways of partitioning V into i independent sets and $\chi(G)$ is the chromatic number. For use later, recall that, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of vertices, their union is defined by the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $P(G_1 \cup G_2, \lambda) = P(G_1, \lambda)P(G_2, \lambda)$, see for example [4, Sec. 1.2]. In this paper, for a given graph H , we present some properties for the families of graphs $O_n \cup H$, $K_n \cup H$ and $T_n \cup H$. In the next section, we give some recurrence relations for the coefficients $\alpha_k(O_n \cup H)$, $\alpha_k(K_n \cup H)$ and $\alpha_k(T_n \cup H)$ and some results on log-concavity and Pólya-frequency for sequences related to these coefficients. In the three last sections, we present three applications on restricted Stirling numbers of the second kind.

2 Recurrence relations and some consequences

Let H be any graph of h vertices, O_n be the graph of n (≥ 1) vertices and no edges and let O_0 be the graph with no vertices, K_n be the complete graph of n (≥ 1) vertices with K_0 be a graph with no vertices and T_n be a tree of n (≥ 1) vertices with T_0 be a graph with no vertices. In this section, we give some recurrence relations for the coefficients $\alpha_k(O_n \cup H)$, $\alpha_k(K_n \cup H)$, $\alpha_k(T_n \cup H)$ and some of their consequences.

Theorem 1 *Let n, s, k be nonnegative integers with $0 \leq s \leq n$. Then, $\alpha_k(O_n \cup H) = 0$ if $k < \chi(H)$ or $k > n + h$ and for $\chi(H) \leq k \leq n + h$ we have*

$$\alpha_k(O_n \cup H) = \sum_{j=\chi(H)}^k \left\{ \begin{matrix} s+j \\ k \end{matrix} \right\}_j \alpha_j(O_{n-s} \cup H).$$

In particular, for $s = n$, we get

$$\alpha_k(O_n \cup H) = \sum_{j=\chi(H)}^k \left\{ \begin{matrix} n+j \\ k \end{matrix} \right\}_j \alpha_j(H),$$

and, for $s = 1$, we get

$$\alpha_k(O_n \cup H) = k\alpha_k(O_{n-1} \cup H) + \alpha_{k-1}(O_{n-1} \cup H), \quad n \geq 1,$$

where the numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r$ are the r -Stirling numbers of the second kind.

Proof. From [4, Sec. 1.2] we have $P(O_n \cup H, \lambda) = \lambda^s P(O_{n-s} \cup H, \lambda)$, $0 \leq s \leq n$. Then, since the graph $O_n \cup H$ is of order $n + h$ and

$$\chi(O_n \cup H) = \max(\chi(O_n), \chi(H)) = \max(1, \chi(H)) = \chi(H),$$

we may state that $\alpha_k(O_n \cup H) = 0$ if $k < \chi(H)$ or $k > n + h$, and otherwise, we have

$$\sum_{k=\chi(H)}^{n+h} \alpha_k(O_n \cup H) (\lambda)_k = \sum_{j=\chi(H)}^{n+h} \alpha_j(O_{n-s} \cup H) \lambda^s (\lambda)_j.$$

On using the known identity $(z + j)^s = \sum_{k=0}^s \left\{ \begin{matrix} s+j \\ k+j \end{matrix} \right\}_j (z)_k$, see [3], we can write

$$\lambda^s (\lambda)_j = (\lambda)_j (\lambda - j + j)^s = \sum_{k=0}^s \left\{ \begin{matrix} s+j \\ k+j \end{matrix} \right\}_j (\lambda)_j (\lambda - j)_k$$

and since $(\lambda)_j (\lambda - j)_k = (\lambda)_{k+j}$ we get $\lambda^s (\lambda)_j = \sum_{k=0}^s \left\{ \begin{matrix} s+j \\ k+j \end{matrix} \right\}_j (\lambda)_{k+j} = \sum_{k=j}^{s+j} \left\{ \begin{matrix} s+j \\ k \end{matrix} \right\}_j (\lambda)_k$. Then

$$\begin{aligned} \sum_{k=\chi(H)}^{n+h} \alpha_k(O_n \cup H) (\lambda)_k &= \sum_{j=\chi(H)}^{n+h} \alpha_j(O_{n-s} \cup H) \sum_{k=j}^{s+j} \left\{ \begin{matrix} s+j \\ k \end{matrix} \right\}_j (\lambda)_k \\ &= \sum_{k=j}^{s+j} (\lambda)_k \sum_{j=\chi(H)}^k \alpha_j(O_{n-s} \cup H) \left\{ \begin{matrix} s+j \\ k \end{matrix} \right\}_j. \end{aligned}$$

So, we get $\alpha_k(O_n \cup H) = \sum_{j=\chi(H)}^k \alpha_j(O_{n-s} \cup H) \left\{ \begin{matrix} s+j \\ k \end{matrix} \right\}_j$. □

Theorem 2 Let n, s, k be nonnegative integers with $0 \leq s \leq n$. Then, $\alpha_k(K_n \cup H) = 0$ if $k < \max(n, \chi(H))$ or $k > n + h$ and for $\max(n, \chi(H)) \leq k \leq n + h$ we have

$$\alpha_k(K_n \cup H) = \sum_{j=\max(n-s, \chi(H))}^k \binom{s}{k-j} (j+s-n)_{s+j-k} \alpha_j(K_{n-s} \cup H).$$

In particular, for $s = n$, we get

$$\alpha_k(K_n \cup H) = \sum_{j=\chi(H)}^k \binom{n}{k-j} (j)_{n+j-k} \alpha_j(H),$$

and, for $s = 1$, we get

$$\alpha_k(K_n \cup H) = (k - n + 1) \alpha_k(K_{n-1} \cup H) + \alpha_{k-1}(K_{n-1} \cup H), \quad n \geq 1.$$

Proof. From [4, Sec. 1.2] we have $P(K_n \cup H, \lambda) = (\lambda - n + 1) P(K_{n-1} \cup H, \lambda)$. Hence

$$P(K_n \cup H, \lambda) = (\lambda - n + s)_s P(K_{n-s} \cup H, \lambda), \quad 0 \leq s \leq n.$$

Then, since the graph $K_n \cup H$ is of order $n + h$ and

$$\chi(K_n \cup H) = \max(\chi(K_n), \chi(H)) = \max(n, \chi(H)),$$

we state that $\alpha_k(K_n \cup H) = 0$ if $k < \max(n, \chi(H))$ or $k > n + h$, and otherwise, we have

$$\sum_{k=\max(n, \chi(H))}^{n+h} \alpha_k(K_n \cup H) (\lambda)_k = \sum_{j=\max(n-s, \chi(H))}^{n-s+h} \alpha_j(K_{n-s} \cup H) (\lambda - n + s)_s (\lambda)_j.$$

By the fact that the sequence of polynomials $((\lambda)_n, n \geq 0)$ is of binomial type, the expression $(\lambda - n + s)_s (\lambda)_j$ can be written as

$$\begin{aligned} (\lambda - n + s)_s (\lambda)_j &= (\lambda - j + j + s - n)_s (\lambda)_j \\ &= \sum_{k=0}^s \binom{s}{k} (j + s - n)_{s-k} (\lambda - j)_k (\lambda)_j \\ &= \sum_{k=0}^s \binom{s}{k} (j + s - n)_{s-k} (\lambda)_{k+j} \\ &= \sum_{k=j}^{s+j} \binom{s}{k-j} (j + s - n)_{s+j-k} (\lambda)_k. \end{aligned}$$

So, it results that

$$\begin{aligned} &\sum_{k=\max(n, \chi(H))}^{n+h} \alpha_k(K_n \cup H) (\lambda)_k \\ &= \sum_{j=\max(n-s, \chi(H))}^{n-s+h} \alpha_j(K_{n-s} \cup H) \sum_{k=j}^{s+j} \binom{s}{k-j} (j + s - n)_{s+j-k} (\lambda)_k \\ &= \sum_{k=\max(n, \chi(H))}^{n+h} (\lambda)_k \sum_{j=\max(n-s, \chi(H))}^k \binom{s}{k-j} (j + s - n)_{s+j-k} \alpha_j(K_{n-s} \cup H). \end{aligned}$$

The identification of the coefficients of $(\lambda)_k$ complete the proof. \square

Theorem 3 Let n, s, k be nonnegative integers with $0 \leq s \leq n$. Then

$$\alpha_k(T_n \cup H) = 0 \quad \text{for } k < \max(2 - \delta_{(n=1)} - 2\delta_{(n=0)}, \chi(H)) \quad \text{or } k > n + h,$$

and, for $\max(2 - \delta_{(n=1)} - 2\delta_{(n=0)}, \chi(H)) \leq k \leq n + h$ we have

$$\alpha_k(T_n \cup H) = \sum_{j=\max(k-s, 0)}^k \left\{ \begin{matrix} s+j-1 \\ k-1 \end{matrix} \right\}_{j-1} \alpha_j(T_{n-s} \cup H).$$

In particular, for $s = n$, we get

$$\alpha_k(T_n \cup H) = \sum_{j=1}^k \left\{ \begin{matrix} s+j-1 \\ k-1 \end{matrix} \right\}_{j-1} \alpha_j(H)$$

and, for $s = 1$, we get

$$\alpha_k(T_n \cup H) = (k-1) \alpha_k(T_{n-1} \cup H) + \alpha_{k-1}(T_{n-1} \cup H), \quad n \geq 1.$$

Proof. The graph $T_n \cup H$ is of order $n + h$ and we have

$$\chi(T_n \cup H) = \max(\chi(T_n), \chi(H)) = \max(2 - \delta_{(n=1)} - 2\delta_{(n=0)}, \chi(H)),$$

we conclude that $\alpha_k(T_n \cup H) = 0$ if $k < \max(2 - \delta_{(n=1)} - 2\delta_{(n=0)}, \chi(H))$ or $k > n + h$, and otherwise, we have

$$P(T_n \cup H, \lambda) = P(T_n, \lambda) P(H, \lambda) = \lambda(\lambda-1)^{n-1} P(H, \lambda),$$

which gives $P(T_n \cup H, \lambda) = (\lambda-1)^s P(T_{n-s} \cup H, \lambda)$, $0 \leq s \leq n$.

This relation is equivalent to

$$\sum_{k=0}^{n+h} \alpha_k(T_n \cup H) (\lambda)_k = \sum_{j=0}^{n+h-s} \alpha_j(T_{n-s} \cup H) (\lambda-1)^s (\lambda)_j.$$

Similarly to $\lambda^s (\lambda)_j$ (see proof of Theorem 1), we get

$$(\lambda-1)^s (\lambda)_j = \sum_{k=j}^{s+j} \left\{ \begin{matrix} s+j-1 \\ k-1 \end{matrix} \right\}_{j-1} (\lambda)_k.$$

Then, the last equality becomes

$$\begin{aligned} \sum_{k=0}^{n+h} \alpha_k(T_n \cup H) (\lambda)_k &= \sum_{j=0}^{n+h-s} \alpha_j(T_{n-s} \cup H) \sum_{k=j}^{s+j} \left\{ \begin{matrix} s+j-1 \\ k-1 \end{matrix} \right\}_{j-1} (\lambda)_k \\ &= \sum_{k=0}^{n+h} (\lambda)_k \sum_{j=\max(k-s, 0)}^k \left\{ \begin{matrix} s+j-1 \\ k-1 \end{matrix} \right\}_{j-1} \alpha_j(T_{n-s} \cup H), \end{aligned}$$

which gives thus the result. \square

To give the following proposition, let us recall some definitions and results on log-concavity, Pólya-frequency and q -log-convexity. Indeed, let u_0, u_1, u_2, \dots , be a sequence of nonnegative real numbers. It is log-concave (LC) if $u_{i-1}u_{i+1} \leq u_i^2$ for all $i > 0$, and, it is called a Pólya-frequency sequence (or a PF sequence) if all minors of the matrix $A = (u_{i-j})_{i,j \geq 0}$ have nonnegative determinants (where $u_k = 0$ if $k < 0$), for more information see [5]. A sequence of real polynomials $(P_n(q), n \geq 0)$ is called

q -log-convex if the polynomial $P_n(q)^2 - P_{n-1}(q)P_{n+1}(q)$ has nonnegative coefficients for all $n \geq 1$, see [12, 13, 19]. Some known results on such sequences are given as follows.

Let $(T(n, k), n, k \geq 0)$ be sequence of nonnegative numbers satisfying the recurrence

$$T(n, k) = (a_1 n + a_2 k + a_3) T(n-1, k) + (b_1 n + b_2 k + b_3) T(n-1, k-1), \quad n \geq k \geq 1,$$

with $T(n, k) = 0$ unless $0 \leq k \leq n$, $T(0, 0) > 0$, $a_1 \geq 0$, $a_1 + a_2 \geq 0$, $a_1 + a_3 \geq 0$ and $b_1 \geq 0$, $b_1 + b_2 \geq 0$, $b_1 + b_2 + b_3 \geq 0$. It is shown in [6, Thm. 2] that, for each fixed n , the sequence $(T(n, k), 0 \leq k \leq n)$ is log-concave. If further we have $a_2 b_1 \geq a_1 b_2$ and $a_2(b_1 + b_2 + b_3) \geq (a_1 + a_3)b_2$, this sequence is Pólya frequency sequence [15, Cor. 3] and further, by setting $T_n(q) = \sum_{k=0}^n T(n, k) q^k$, if

$$(a_2 b_1 - a_1 b_2) n + a_2 b_2 k + a_2 b_3 - a_3 b_2 \geq 0 \text{ for } 0 < k \leq n,$$

then, the sequence of polynomials $(T_n(q), n \geq 0)$ is q -log-convex [8, Thm. 4.1].

Proposition 4 *Let $(U(n, k), n, k \geq 0)$, $(V(n, k), n, k \geq 0)$, and $(W(n, k), n, k \geq 0)$, be sequences of nonnegative numbers with $U(n, k) = V(n, k) = W(n, k) = 0$ when $k > n$ and for $0 \leq k \leq n$, these sequences are defined by*

$$\begin{aligned} U(n, k) &= \alpha_{k+h}(O_n \cup H), \\ V(n, k) &= \alpha_{k+h}(T_n \cup H), \\ W(n, k) &= \alpha_{k+h}(K_n \cup H) \end{aligned}$$

and let

$$\begin{aligned} U_n(q) &= \sum_{k=0}^n U(n, k) q^k, \\ V_n(q) &= \sum_{k=0}^n V(n, k) q^k. \end{aligned}$$

Then, the sequences $(U(n, k), 0 \leq k \leq n)$ and $(V(n, k), 0 \leq k \leq n)$ are log-concave and Pólya frequency sequences, the sequences of real polynomials $(U_n(q), n \geq 0)$ and $(V_n(q), n \geq 0)$ are q -log-convex sequences and the sequence $(W(n, k), 0 \leq k \leq n)$ is a Pólya frequency sequence.

Proof. We have $U(0, 0) = V(0, 0) = W(0, 0) = \alpha_h(H) = 1$ and for $n \geq 1$, Theorems 1, 2 and 3 imply

$$\begin{aligned} U(n, k) &= U(n-1, k-1) + (k+h) U(n-1, k), \\ V(n, k) &= V(n-1, k-1) + (k+h-1) V(n-1, k), \\ W(n, k) &= W(n-1, k-1) + (k-n+h+1) W(n-1, k). \end{aligned}$$

So, the log-concavity follows from [6, Thm. 2], Pólya frequency follows from [15, Cor. 3] and q -log-convexity follows from [8, Thm. 4.1]. \square

3 Application to the graph $K_{r_1} \cup \dots \cup K_{r_p} \cup O_n$

Let $p \geq 1$ be an integer. We consider the graph $G_{n, \mathbf{r}_p} = K_{r_1} \cup \dots \cup K_{r_p} \cup O_n$ of order $n + r_1 + \dots + r_p$ and chromatic number

$$\chi(G_{n, \mathbf{r}_p}) = \max(\chi(K_{r_1}), \dots, \chi(K_{r_p}), \chi(O_n)) = \max(r_1, \dots, r_p, 1).$$

First of all, recall the definition of the (r_1, \dots, r_p) -Stirling number of the second kind introduced by Mihoubi et al. in [10, 9].

Definition 5 Let R_1, \dots, R_p be subsets of the set $[n]$ with $|R_i| = r_i$ and $R_i \cap R_j = \emptyset$ for all $i, j = 1, \dots, p, i \neq j$. The (r_1, \dots, r_p) -Stirling number of the second kind, $p \geq 1$, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_1, \dots, r_p}$, counts the number of partitions of the set $[n] := \{1, 2, \dots, n\}$ into k non-empty subsets such that the elements of each of the p sets R_1, \dots, R_p are in distinct subsets.

From this definition, one can see easily that these numbers satisfy

$$\begin{aligned} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_1, \dots, r_p} &= 0, \quad n < r_1 + \dots + r_p \text{ or } k < \max(r_1, \dots, r_p), \\ \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_1, \dots, r_p} &= \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_p} \text{ if } r_1, \dots, r_{p-1} \in \{0, 1\}, \\ \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_1, \dots, r_p} &= \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r_{\sigma(1)}, \dots, r_{\sigma(p)}} \text{ for all permutations } \sigma \text{ on the set } \{1, \dots, p\}. \end{aligned}$$

Furthermore, for $r_1 \leq r_2 \leq \dots \leq r_p$, the coefficient $\alpha_k(G_{n, \mathbf{r}_p})$ represents the number of ways of partitioning the set of $n + |\mathbf{r}_p|$ vertices of G_{n, \mathbf{r}_p} into k independent subsets, and by the definition of the graph G_{n, \mathbf{r}_p} , the elements of each of the p subgraphs K_{r_1}, \dots, K_{r_p} must be in distinct subsets. So, this number is exactly $\left\{ \begin{smallmatrix} n + |\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{\mathbf{r}_p}$. Then, we may state that

$$\alpha_k(G_{n, \mathbf{r}_p}) = \left\{ \begin{smallmatrix} n + |\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{\mathbf{r}_p}.$$

In particular

$$\alpha_k(K_r \cup O_n) = \left\{ \begin{smallmatrix} n + r \\ k \end{smallmatrix} \right\}_r \text{ and } \alpha_k(O_n) = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}.$$

Now, we present new proofs for some properties of the (r_1, \dots, r_p) -Stirling numbers of the second kind and other results.

Theorem 6 For $r_1 \leq r_2 \leq \dots \leq r_p$, the polynomial $(z + r_p)_{r_1} \dots (z + r_p)_{r_{p-1}} (z + r_p)^n$ can be written in the basis $\{(\lambda)_k, k = 0, 1, \dots, n + |\mathbf{r}_{p-1}|\}$ as follows

$$(z + r_p)_{r_1} \dots (z + r_p)_{r_{p-1}} (z + r_p)^n = \sum_{k=0}^{n + |\mathbf{r}_{p-1}|} \left\{ \begin{smallmatrix} n + |\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{\mathbf{r}_p} (z)_k,$$

where $\mathbf{r}_p := (r_1, \dots, r_p)$ and $|\mathbf{r}_p| := r_1 + \dots + r_p$. In particular, we have

$$(z + r)^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n + r \\ k + r \end{smallmatrix} \right\}_r (z)_k.$$

Proof. From [4, Sec. 1.2], the chromatic polynomial of the graph G_{n, \mathbf{r}_p} is

$$\begin{aligned} P(G_{n, \mathbf{r}_p}, z + r_p) &= P(O_n, z + r_p) P(K_{r_1}, z + r_p) \dots P(K_{r_p}, z + r_p) \\ &= (z + r_p)^n (z + r_p)_{r_1} (z + r_p)_{r_2} \dots (z + r_p)_{r_p}, \end{aligned}$$

and, by definition of the chromatic polynomial, we have

$$\begin{aligned} P(G_{n, \mathbf{r}_p}, z + r_p) &= \sum_{k=0}^{n + |\mathbf{r}_{p-1}|} \left\{ \begin{smallmatrix} n + |\mathbf{r}_p| \\ k + r_p \end{smallmatrix} \right\}_{\mathbf{r}_p} (z + r_p)_{k + r_p} \\ &= \sum_{k=0}^{n + |\mathbf{r}_{p-1}|} \left\{ \begin{smallmatrix} n + |\mathbf{r}_p| \\ k + r_p \end{smallmatrix} \right\}_{\mathbf{r}_p} (z + r_p)_{r_p} (z)_k. \end{aligned}$$

which complete the proof. \square

Remark 7 From the following identity (see [14] or [4, pp. 102]) we have

$$\alpha_k(G_{n, \mathbf{r}_p}) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} P(G_{n, \mathbf{r}_p}, j),$$

or equivalently,

$$\begin{aligned} \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + r_p \end{matrix} \right\}_{\mathbf{r}_p} &= \frac{1}{(k + r_p)!} \sum_{j=r_p}^{k+r_p} (-1)^{k+r_p-j} \binom{k+r_p}{j} (j)_{r_1} \cdots (j)_{r_p} j^n \\ &= \frac{1}{(k + r_p)!} \sum_{j=0}^k (-1)^{k-j} \binom{k+r_p}{j+r_p} (j+r_p)_{r_1} \cdots (j+r_p)_{r_p} (j+r_p)^n \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (j+r_p)_{r_1} \cdots (j+r_p)_{r_{p-1}} (j+r_p)^n. \end{aligned}$$

So, the two initial values of these numbers are given by

$$\begin{aligned} \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ r_p \end{matrix} \right\}_{\mathbf{r}_p} &= (r_p)^n (r_p)_{r_1} \cdots (r_p)_{r_{p-1}}, \\ \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ r_p + 1 \end{matrix} \right\}_{\mathbf{r}_p} &= (r_p + 1)_{r_1} \cdots (r_p + 1)_{r_{p-1}} (r_p + 1)^n - (r_p)_{r_1} \cdots (r_p)_{r_{p-1}} (r_p)^n. \end{aligned}$$

Now, set $K_{r_i} \cup H := K_{r_1} \cup \cdots \cup K_{r_p} \cup O_n$ ($i = 1, \dots, p$) in Theorem 2 to obtain:

Corollary 8 Let n , s and k be nonnegative integers with $n \geq |\mathbf{r}_p|$. Then, for a fixed i in the set $\{1, \dots, p\}$, if $r_i \geq 1$ we have

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathbf{r}_p} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_{\mathbf{r}_p - \mathbf{e}_i} + (k+1-r_i) \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{\mathbf{r}_p - \mathbf{e}_i},$$

where \mathbf{e}_i denote the i -th vector of the canonical basis of \mathbb{R}^p .

A particular case of the last corollary is when $p = 1$ we obtain the following known identity

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_{r-1} + (k+1-r) \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{r-1}.$$

By setting $H := K_{r_1} \cup \cdots \cup K_{r_p}$ in Theorem 1 we obtain:

Corollary 9 Let n , k be integers such that $r_p \leq k \leq n$ and $n \geq |\mathbf{r}_p|$. We have

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathbf{r}_p} = \sum_{j=r_p}^k \left\{ \begin{matrix} s+j \\ k \end{matrix} \right\}_j \left\{ \begin{matrix} n-s \\ j \end{matrix} \right\}_{\mathbf{r}_p}, \quad 0 \leq s \leq n - |\mathbf{r}_p|.$$

In particular, for $s = 1$, these numbers obey to the following recurrence relation:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathbf{r}_p} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_{\mathbf{r}_p} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{\mathbf{r}_p},$$

and for $s = n - |\mathbf{r}_p|$ we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\mathbf{r}_p} = \sum_{j=r_p}^k \left\{ \begin{matrix} n - |\mathbf{r}_p| + j \\ k \end{matrix} \right\}_j \left\{ \begin{matrix} |\mathbf{r}_p| \\ j \end{matrix} \right\}_{\mathbf{r}_p}.$$

In particular, for $p = s = 1$ in Corollary 9, we obtain the known identity

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_r + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_r,$$

and for $s = n - |\mathbf{r}_p|$, $p = 2$ and $(r_1, r_2) = (1, r)$ in Corollary 9, we obtain

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{1,r} = (r+1) \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_r + \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r+1}.$$

Now, upon using Proposition 4, we may state the following corollary.

Corollary 10 For $0 \leq k \leq n$ let

$$U(n, k) = \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{\mathbf{r}_p}.$$

Then, the sequence $(U(n, k), 0 \leq k \leq n)$ is log-concave and Pólya frequency sequence, and, the sequence of polynomials $(U_n(q), n \geq 0)$ defined by

$$U_n(q) = \sum_{k=0}^n \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{\mathbf{r}_p} q^k$$

is a q -log-convex sequence.

4 Application to the graph $K_{r_1, \dots, r_p} \cup O_n$

Let $p \geq 2$ be an integer. For a second application of the chromatic polynomials, we consider in this section the graph $K_{n, \mathbf{r}_p} = K_{r_1, \dots, r_p} \cup O_n$ of order $n + r_1 + \dots + r_p$ and chromatic number

$$\chi(K_{n, \mathbf{r}_p}) = \max(\chi(K_{r_1, \dots, r_p}), \chi(O_n)) = \max(p, 1) = p.$$

Similarly to the numbers (r_1, \dots, r_p) -Stirling numbers, we present here a new class of the Stirling numbers having also triangular recurrence relation.

To start, let us giving the following definition.

Definition 11 Let R_1, \dots, R_p be subsets of the set $[n]$ with $|R_i| = r_i$ and $R_i \cap R_j = \emptyset$ for all $i, j = 1, \dots, p$, $i \neq j$. The $K(r_1, \dots, r_p)$ -Stirling number of the second kind, denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(r_1, \dots, r_p)}$, counts the number of partitions of the set $[n]$ into k non-empty subsets such that if $x \in R_i$ and $y \in R_j$ with $i \neq j$, then each subset do not containing simultaneously x and y .

From this definition, we may state the following:

$$\begin{aligned} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(r_1, \dots, r_p)} &= 0, \quad n < r_1 + \dots + r_p \text{ or } k < p, \\ \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(r_1, \dots, r_p)} &= \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r_1 + \dots + r_p} \quad \text{if } r_1, \dots, r_p \in \{0, 1\}, \\ \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(r_1, \dots, r_p)} &= \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(r_{\sigma(1)}, \dots, r_{\sigma(p)})} \quad \text{for all permutations } \sigma \text{ on the set } \{1, \dots, p\}. \end{aligned}$$

Furthermore, for $r_1 \leq r_2 \leq \dots \leq r_p$, the coefficient $\alpha_k(K_{n, \mathbf{r}_p})$ represents the number of ways of partitioning the set of $n + |\mathbf{r}_p|$ vertices of K_{n, \mathbf{r}_p} into k independent sets, and by the definition of the graph

K_{n, \mathbf{r}_p} , any two elements x of the i -th block of the subgraph K_{r_1, \dots, r_p} and y of the j -th block of K_{r_1, \dots, r_p} , with $i \neq j$, can't be in the same subset. So, this number is exactly $\left\{ \begin{smallmatrix} n+r_1+\dots+r_p \\ k \end{smallmatrix} \right\}_{K(r_1, \dots, r_p)}$. Then, we may state that

$$\alpha_k(K_{n, \mathbf{r}_p}) = \left\{ \begin{smallmatrix} n+r_1+\dots+r_p \\ k \end{smallmatrix} \right\}_{K(r_1, \dots, r_p)} := \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{K(\mathbf{r}_p)}.$$

Set $H := K_{r_1, \dots, r_p}$ in Theorem 1 to obtain:

Corollary 12 *Let n, k be integers such that $p \leq k \leq n$ and $n \geq |\mathbf{r}_p|$. We have*

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} = \sum_{j=p}^k \left\{ \begin{smallmatrix} s+j \\ k \end{smallmatrix} \right\}_j \left\{ \begin{smallmatrix} n-s \\ j \end{smallmatrix} \right\}_{K(\mathbf{r}_p)}, \quad 0 \leq s \leq n - |\mathbf{r}_p|.$$

For $s = 1$ we conclude that these numbers obey to the recurrence relation:

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}_{K(\mathbf{r}_p)}, \quad n \geq k \geq 1.$$

and for $s = n - |\mathbf{r}_p|$ in Corollary 12, we get

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} = \sum_{j=p}^k \left\{ \begin{smallmatrix} n-|\mathbf{r}_p|+j \\ k \end{smallmatrix} \right\}_j \left\{ \begin{smallmatrix} |\mathbf{r}_p| \\ j \end{smallmatrix} \right\}_{K(\mathbf{r}_p)}.$$

In particular, for $s = r_1 = \dots = r_p = 1$ in Corollary 12 we obtain the known identity:

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_p = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_p + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}_p.$$

Remark 13 *By setting $s = 1$ and $k = p$ or $p+1$ in Corollary 12 and upon using the identities given in [20] (see also [4, Lemma 4.4.2]) by*

$$\left\{ \begin{smallmatrix} |\mathbf{r}_p| \\ p \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} = 1, \quad \left\{ \begin{smallmatrix} |\mathbf{r}_p| \\ p+1 \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} = \sum_{j=1}^p 2^{r_j-1} - p,$$

we obtain the two initial values of these numbers

$$\begin{aligned} \left\{ \begin{smallmatrix} n \\ p \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} &= \left\{ \begin{smallmatrix} n-|\mathbf{r}_p|+p \\ p \end{smallmatrix} \right\}_p, \\ \left\{ \begin{smallmatrix} n \\ p+1 \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} &= \left\{ \begin{smallmatrix} n-|\mathbf{r}_p|+p \\ p+1 \end{smallmatrix} \right\}_p + \left\{ \begin{smallmatrix} n-|\mathbf{r}_p|+p+1 \\ p+1 \end{smallmatrix} \right\}_{p+1} \left(\sum_{j=1}^p 2^{r_j-1} - p \right). \end{aligned}$$

Proposition 14 *Let*

$$B(\lambda; K_{n, \mathbf{r}_p}) := \sum_{k \geq 0} \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{K(\mathbf{r}_p)} \lambda^k.$$

Then, we have

$$\begin{aligned} B(\lambda; K_{n, \mathbf{r}_p}) &= \lambda \exp(-\lambda) \frac{d}{d\lambda} (\exp(\lambda) B(\lambda; K_{n-1, \mathbf{r}_p})), \quad n \geq 1, \\ B(\lambda; K_{0, \mathbf{r}_p}) &= B_{r_1}(\lambda) \cdots B_{r_p}(\lambda), \end{aligned}$$

where $B_n(\lambda) = \sum_{j=0}^n \left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\} \lambda^j$ is the classical Bell polynomial.

Proof. From Corollary 12 we have

$$\sum_{k \geq 1} \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k \end{matrix} \right\}_{K(\mathbf{r}_p)} \lambda^k = \sum_{k \geq 1} k \left\{ \begin{matrix} n + |\mathbf{r}_p| - 1 \\ k \end{matrix} \right\}_{K(\mathbf{r}_p)} \lambda^k + \sum_{k \geq 1} \left\{ \begin{matrix} n + |\mathbf{r}_p| - 1 \\ k - 1 \end{matrix} \right\}_{K(\mathbf{r}_p)} \lambda^k$$

which gives the first identity. The second one follows from [4, Lemma 4.4.1]. \square

Corollary 15 For $n \geq |\mathbf{r}_p|$, the numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(\mathbf{r}_p)}$, $k = p, p+1, \dots, n$, are log-concave.

Proof. Apply Role's Theorem on the function $f_n(\lambda) := \exp(\lambda) B(\lambda; K_{n, \mathbf{r}_p})$ and use the fact that the roots of the polynomial $B_n(\lambda)$ are real non-positive (see for example [15]) to conclude (by induction on n) that the polynomial $B(\lambda; K_{n, \mathbf{r}_p})$ has only real non-positive roots. After that, apply Newton's inequality [7, p. 52] to complete the proof. \square Similarly to Corollary 10, Proposition 4 states that we have

Corollary 16 For $0 \leq k \leq n$ let

$$W(n, k) = \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{K(\mathbf{r}_p)}.$$

Then, the sequence $(W(n, k), 0 \leq k \leq n)$ is a Pólya frequency sequence.

5 Application to the graph $T_{r_1} \cup \dots \cup T_{r_p} \cup O_n$

Let $p \geq 1$ be an integer. To give a third application of the chromatic polynomials, we consider in this section the graph $T_{n, \mathbf{r}_p} = T_{r_1} \cup \dots \cup T_{r_p} \cup O_n$. Here

$$\chi(T_{n, \mathbf{r}_p}) = \max(\chi(T_{r_1}), \dots, \chi(T_{r_p}), \chi(O_n)) = \min(r_1 \cdots r_p, 2).$$

Similarly to the above numbers, we present here a new class of the Stirling numbers having also triangular recurrence relation. Then, we may start by giving the following definition.

Definition 17 Let R_1, \dots, R_p be subsets of the set $[n]$ with $|R_i| = r_i$ and $R_i \cap R_j = \emptyset$ for all $i, j = 1, \dots, p$, $i \neq j$. The $T(r_1, \dots, r_p)$ -Stirling number of the second kind, denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{T(r_1, \dots, r_p)}$, counts the number of partitions of the set $[n]$ into k non-empty subsets such that the minimum element of R_i can't be in the same subset with any other element of R_i , $i = 1, \dots, p$.

From this definition, we may state the following:

$$\begin{aligned} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{T(r_1, \dots, r_p)} &= 0, \quad n < r_1 + \dots + r_p \text{ or } k < \min(r_1 \cdots r_p, 2), \\ \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{T(r_1, \dots, r_p)} &= \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \text{ if } r_1, \dots, r_p \in \{0, 1\}, \\ \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{T(r_1, \dots, r_p)} &= \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{K(r_{\sigma(1)}, \dots, r_{\sigma(p)})} \text{ for all permutations } \sigma \text{ on the set } \{1, \dots, p\}. \end{aligned}$$

Furthermore, since the trees of same order have the same chromatic polynomial, it is sufficient to apply such results on the star graphs. So, choice T_i to be the star graph with r_i vertices ($T_i =$

K_{1,r_i-1}) with the universal vertex is the minimum element of R_i (the set-vertex of T_i). Hence, the coefficient $\alpha_k(T_{n,\mathbf{r}_p})$ represents the number of ways of partitioning the set of $n + |\mathbf{r}_p|$ vertices of T_{n,\mathbf{r}_p} into k independent sets, and by the definition of the graph T_{n,\mathbf{r}_p} , the universal vertex of T_{r_i} ($i = 1, \dots, p$) can't be in the same independent set with any other vertex of T_{r_i} . So, this number is exactly $\left\{ \begin{smallmatrix} n+r_1+\dots+r_p \\ k \end{smallmatrix} \right\}_{T(r_1,\dots,r_p)}$.

Then, we may state that

$$\alpha_k(T_{n,\mathbf{r}_p}) = \left\{ \begin{smallmatrix} n+r_1+\dots+r_p \\ k \end{smallmatrix} \right\}_{T(r_1,\dots,r_p)} := \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)}.$$

Theorem 18 For $1 \leq r_1 \leq r_2 \leq \dots \leq r_p$, the polynomial $z^{n+p}(z-1)^{|\mathbf{r}_p|-p}$ can be written in the basis $\{(\lambda)_k, k = 0, 1, \dots, n + |\mathbf{r}_p|\}$ as follows:

$$z^{n+p}(z-1)^{|\mathbf{r}_p|-p} = \sum_{k=0}^{n+|\mathbf{r}_p|} \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} (z)_k,$$

Furthermore, we have

$$\left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} = \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ k \end{smallmatrix} \right\}_{T(|\mathbf{r}_p|-p+1)}.$$

Proof. From [4, Sec. 1.2], the chromatic polynomial of the graph T_{n,\mathbf{r}_p} is

$$\begin{aligned} P(T_{n,\mathbf{r}_p}, z) &= P(O_n, z) P(T_{r_1}, z) \cdots P(T_{r_p}, z) \\ &= (z)^{n+p} (z-1)^{|\mathbf{r}_p|-p} \\ &= \sum_{i=0}^{n+|\mathbf{r}_p|} \left\{ \begin{smallmatrix} n+|\mathbf{r}_p| \\ i \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} (z)_i. \end{aligned}$$

This gives the first result. In particular, we have

$$z^{n+1}(z-1)^{r-1} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n+r \\ k \end{smallmatrix} \right\}_{T(r)} (z)_k,$$

and by combining this result with the first one we obtain the second result. \square

For the choice $H := T_{r_1} \cup \dots \cup T_{r_p}$ in Theorem 1 we get:

Corollary 19 Let n, k be integers such that $2 \leq k \leq n$ and $n \geq |\mathbf{r}_p|$. Then, for $1 \leq r_1 \leq \dots \leq r_p$, we have

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} = \sum_{j=2}^k \left\{ \begin{smallmatrix} s+j \\ k \end{smallmatrix} \right\}_j \left\{ \begin{smallmatrix} n-s \\ j \end{smallmatrix} \right\}_{T(\mathbf{r}_p)}, \quad 0 \leq s \leq n - |\mathbf{r}_p|.$$

In particular, for $s = 1$, these numbers obey to the following triangular recurrence relation:

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} = \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} + k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)},$$

and for $s = n - |\mathbf{r}_p|$ we get

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{T(\mathbf{r}_p)} = \sum_{j=2}^k \left\{ \begin{smallmatrix} n-|\mathbf{r}_p|+j \\ k \end{smallmatrix} \right\}_j \left\{ \begin{smallmatrix} |\mathbf{r}_p| \\ j \end{smallmatrix} \right\}_{T(\mathbf{r}_p)}.$$

For $i = 1, \dots, p$, set $s = 1$ and $T_{r_i} \cup H := T_{r_1} \cup \dots \cup T_{r_p} \cup O_n$ in Theorem 3 to obtain:

Corollary 20 *Let n, k be integers such that $1 \leq k \leq n$ and $n \geq |\mathbf{r}_p|$. Then, for a fixed i ($i = 1, \dots, p$) with $r_i \geq 1$, we have*

$$\left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k \end{matrix} \right\}_{T(\mathbf{r}_p)} = (k-1) \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k \end{matrix} \right\}_{T(\mathbf{r}_p - \mathbf{e}_i)} + \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k-1 \end{matrix} \right\}_{T(\mathbf{r}_p - \mathbf{e}_i)}, \quad n \geq 1.$$

Remark 21 *For $1 \leq r_1 \leq \dots \leq r_p$, then similarly to Remark 7, we get*

$$\left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k \end{matrix} \right\}_{T(\mathbf{r}_p)} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^{n+p} (j-1)^{|\mathbf{r}_p|-p},$$

and since $\chi(T_{n, \mathbf{r}_p}) = 2$, when $r_1 \dots r_p > 1$, then the two initial values of these numbers are given by

$$\left\{ \begin{matrix} n + |\mathbf{r}_p| \\ 2 \end{matrix} \right\}_{T(\mathbf{r}_p)} = 2^{n+p-1}, \quad \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ 3 \end{matrix} \right\}_{T(\mathbf{r}_p)} = 3^{n+p-1} \times 2^{|\mathbf{r}_p|-p-1} - 2^{n+p-1},$$

and for $r_1 = \dots = r_p = 1$ we get

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{T(\mathbf{r}_p)} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

Similarly to Corollary 10, we have:

Corollary 22 *For $0 \leq k \leq n$ let*

$$V(n, k) = \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{T(\mathbf{r}_p)}.$$

Then, the sequence $(V(n, k), 0 \leq k \leq n)$ is log-concave and Pólya frequency sequence, and, the sequence of polynomials $(V_n(q), n \geq 0)$ defined by

$$V_n(q) = \sum_{k=0}^n \left\{ \begin{matrix} n + |\mathbf{r}_p| \\ k + |\mathbf{r}_p| \end{matrix} \right\}_{T(\mathbf{r}_p)} q^k$$

is a q -log-convex sequence.

References

- [1] G. D. Birkhoff, A determinant formula for the number of ways of colouring a map. *Ann. Math.*, 14(2):42–46, 1912.
- [2] G. D. Birkhoff and D.C. Lewis, Chromatic polynomials. *Trans. Amer. Math. Soc.* 60:355–451, 1946.
- [3] A. Z. Broder, The r -Stirling numbers. *Discrete Math.* 49: 241–259, 1984.
- [4] F. M. Dong, K. M. Koh and K. L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*. World Scientific, 2005.
- [5] S. Karlin, *Total Positivity, vol.I*. Stanford University Press, Stanford, 1968.

- [6] D. C. Kurtz, A note on concavity properties of triangular arrays of numbers. *J. Combin. Theory Ser. A* 13:135–139, 1972.
- [7] G. H. Hardy, J. E. Littlewood and G. Plóya, *Inequalities* (Cambridge: The University Press), 1952.
- [8] L. L. Liu and Y. Wang, On the log-convexity of combinatorial sequences. *Adv. in Appl. Math.* 39:453–476, 2007.
- [9] M. S. Maamra and M. Mihoubi, The (r_1, \dots, r_p) -Bell polynomials. Preprint available at <http://arxiv.org/abs/1212.3191v1>.
- [10] M. Mihoubi and M. S. Maamra, The (r_1, \dots, r_p) -Stirling numbers of the second kind. *Integers* 12: Article A35, 2012.
- [11] R. C. Read, An introduction to chromatic polynomials. *J. of combinatorial theory* 4:52-71, 1968.
- [12] B. E. Sagan, Log concave sequences of symmetric functions and analogs of the Jacobi–Trudi determinants. *Trans. Amer. Math. Soc.* 329:795–811, 1992.
- [13] B. E. Sagan, Inductive proofs of q -log concavity. *Discrete Math.* 99:298–306, 1992.
- [14] Tomescu, *Problems in combinatorics and graph theory*. Translated from the Romanian by Robert A. Milder. Wiley Interscience Series in Discrete Mathematics. A Wiley Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1985.
- [15] Y. Wang and Y-N. Yeh, Polynomials with real zeros and Pólya frequency sequences. *J. Combin. Theory Ser. A* 109:63–74, 2005.
- [16] E. G. Whitehead, Stirling number identities from chromatic polynomials. *J. Combin. Theory Ser. A* 24:314–317, 1978.
- [17] H. Whitney, A logical expansion mathematics. *Bull. amer. Math. Soc.* 38:572-579, 1932.
- [18] H. Whitney, On the colouring of graphs. *Ann. Math. (12)* 33:688-718, 1933.
- [19] F. Z. Zhao, On log-concavity of a class of generalized Stirling numbers. *Electron. J. Comb.* 19(2):P11, 2012.
- [20] H. W. Zou, The chromatic uniqueness of certain complete t -partite graphs. *Discrete Math.* 275:375–383, 2004.